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Nonlinear H_∞ Optimal Control for Agile Missiles

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Nonlinear H_∞ optimal control is used to design a pitch plane flight control system for a high-angle-of-attack agile missile. The nonlinear H_∞ control law is obtained by approximating the solution to the Hamilton–Jacobi–Isaacs equation. The solution to the partial differential equation is formed using the method of characteristics and is numerically approximated using successive approximations. Nonlinear simulation results using the nonlinear control law are presented. Simulation results testing the algorithms demonstrated excellent performance.

Introduction

NEW interests in missile alternate controls (reaction jets, thrust vectoring) to augment, or possibly eliminate, aerodynamic control surfaces poses a considerable control system design challenge. Low-cost reaction jets are constant thrust devices that result in bang-bang type controls. Thrust vector control actuation systems have hard angle and command magnitude limits and actuator nonlinearities. Blending these alternate controls with aerodynamic control surfaces combine current linear autopilot design problems, solved using linear robust control methods, with nonlinear controls in which design methodologies do not exist. This research in nonlinear control addresses these difficult design problems by proposing a nonlinear H_∞ solution[1–3] to these flight control design problems.

The state space solution of linear H_∞ optimal control problems can be found in Doyle et al.[4] This same problem of reducing the H_∞ norm of a closed-loop system has been viewed as a two person, zero sum, differential game in Basar and Bernhard,[5] where the solution is related to certain algebraic Riccati equations. This approach, for nonlinear systems has been pursued by Ball and Helton[6] (also by Basar and Bernhard[5]). For nonlinear systems, the Riccati equation is replaced with a particular Hamilton–Jacobi equation known as the Isaacs[7] equations [Ref. 7, Eq. (4.3)]. This type of optimal control is referred to as nonlinear H_∞ (L_2-gain) optimal control.[1–3,8]

Consider a nonlinear system modeled by the equations of the form

$$\begin{aligned} \dot{x} &= f(x) + g_1(x)u + g_2(x)w \\ z &= h(x) + d_1(x)u + d_2(x)w \end{aligned} \tag{1}$$

The first equation describes a plant with state x, defined on a neighborhood X of the origin in R^n with control input $u \in R^m$ and subject to a set of exogenous input variables $w \in R^r$ that includes commands to be tracked and/or disturbances to be rejected. The second equation defines the regulated variables $z \in R^s$ that may include tracking errors, for instance, the difference between the actual plant output and its desired reference behavior, expressed as a function of some of the exogenous variables w, as well as a cost of the input u needed to achieve the prescribed control goal. The mappings $f(x)$, $g_i(x)$, $h(x)$, $d_i(x)$, in Eq. (1) are smooth mappings defined in a neighborhood of the origin in R^n. Also, it is assumed that $f(0) = 0$ and $h(0) = 0$.

The purpose of the control is twofold: to achieve closed-loop stability and to attenuate the influence of the exogenous input w on the regulated variable z. A controller that locally asymptotically stabilizes the equilibrium $x = 0$ of the closed-loop system is said to be an admissible controller. The requirement of disturbance attenuation may be dealt with in several different manners, depending on the specific class of exogenous signals to be considered and/or the performance criteria chosen to evaluate the regulated variables. The following characterization taken from Isidori and Astolfi[3] is considered here. Given a positive real number γ, it is said that the exogenous signals are locally attenuated by γ if there exists a neighborhood U of the point $x = 0$ such that for every $T > 0$ and for every piecewise continuous function $w : [0, T] \rightarrow R^r$ for which the initial state $x(0) = \mathbf{0}$ remains in U for all $t \in [0, T]$, the response $z : [0, T] \rightarrow R^s$ of Eq. (1) satisfies

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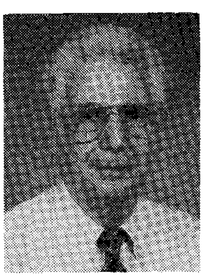

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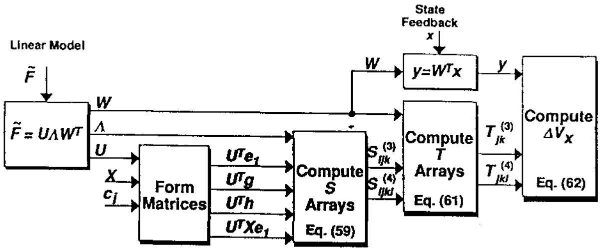

Fig. 1 Calculations performed to compute the nonlinear control.

$$\int_0^T z^T(\tau)z(\tau)\,\mathrm{d}\tau \le \gamma^2 \int_0^T w^T(\tau)w(\tau)\,\mathrm{d}\tau \qquad (2)$$

The problem of local disturbance attenuation with internal stability is to find an admissible controller yielding local attenuation of the exogenous inputs.

In this paper, a solution approach is presented to the Hamilton–Jacobi–Isaacs (HJI) equation arising in nonlinear H_∞ state feedback optimal control problems. This solution approach is applied here to an agile missile (antiair) flight control problem. Only planar dynamics are considered. The goal is to design a nonlinear control that will stabilize the missile and track guidance commands. The evaluate the control algorithms, an agile turn to the rear hemisphere was simulated. A MATRIXx simulation of the missile's dynamics was used. The missile under study, shown in Fig. 1, blends together aerodynamic controls and thrust vector controls. To quickly perform this turn to the rear hemisphere large angles of attack (AOA) are required.

The paper is organized as follows. The second section presents the approach used to solve the HJI nonlinear partial differential equation (PDE). For a derivation of this PDE, see Refs. 1–3, 6, and 8. This section also presents a proof showing that the successive approximation solution approach provides a local contraction mapping. The third section presents the nonlinear missile dynamics and the algorithms implementing the successive approximation solution approach. The fourth section presents nonlinear simulation results. The fifth and final section summarizes the results and presents conclusions and directions for future work in this area.

Approximate Solution of the HJI Equation

Consider the problem of regulating the state $\boldsymbol{x}$ to $\boldsymbol{x} = 0$ by means of a state feedback control law $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$. The design model is given by Eq. (1).

The HJI PDE can be expressed as

$$V_x^* f + h^T h - \begin{bmatrix} \frac{1}{2} g_1 V_x^{*T} + d_1 h \\ \frac{1}{2} g_2 V_x^{*T} + d_2 h \end{bmatrix}^T R^{-1} \begin{bmatrix} \frac{1}{2} g_1 V_x^{*T} + d_1 h \\ \frac{1}{2} g_2 V_x^{*T} + d_2 h \end{bmatrix} = 0 \qquad (3)$$

where the dependency on x has been dropped to shorten the expression, and

$$R = \begin{bmatrix} d_1^T(x)d_1(x) & d_1^T(x)d_2(x) \\ d_2^T(x)d_1(x) & d_2^T(x)d_2(x) - \gamma^2 I \end{bmatrix}$$

The solution approach used here forms the nonlinear H_∞ control law around a gain scheduled linear H_∞ solution, based on the linearized dynamics, and then adds to the linear control law based on an approximate solution to the HJI PDE Eq. (3) using the method of successive approximations. This represents a new approach to solving HJI PDEs.

Equation (3) can be written as

$$0 = h^T h - h^T S R^{-1} S^T h + V_x^*(f - B R^{-1} S^T h)$$
$$- \tfrac{1}{4} V_x^* B R^{-1} B^T V_x^{*T} \qquad (4)$$

where $B = [g_1(x)\ \ g_2(x)]$, $S = [d_1(x)\ \ d_2(x)]$, and γ in $\boldsymbol{R}$ results from a linear H_∞ design using the linearized dynamics about $\boldsymbol{x} = 0$.

The state feedback control is given by

$$u = [1 \quad 0](-R^{-1})\left(\tfrac{1}{2} B^T V_x^{*T} + S^T h\right) \qquad (5)$$

where the Lyapunov function $V^*(x)$ must satisfy the HJI PDE, Eq. (3).

An important special case of the nonlinear dynamics, Eq. (1), useful in aerospace applications is where the nonlinearities are confined only to $\boldsymbol{f}(\boldsymbol{x})$. For this case

$$h(x) = Cx, \qquad g_1(x) = G_1, \qquad g_2(x) = G_2$$
$$d_1(x) = D_1, \qquad d_2(x) = D_2 \qquad (6)$$

The nonlinearities in $\boldsymbol{f}(\boldsymbol{x})$ are modeled as

$$f(x) = Ax + \Delta f(x) \qquad (7)$$

where $\Delta f(x) = \mathcal{O}(x^2)$.

The solution to the linear H_∞ problem is obtained from the following algebraic Riccati equation (ARE):

$$X\tilde{A} + \tilde{A}^T X + \tilde{Q} + X\tilde{R}X = 0 \qquad (8)$$

where

$$B = [G_1 \quad G_2], \qquad S = [D_1 \quad D_2]$$
$$\tilde{Q} = C^T(I - SR^{-1}S^T)C, \qquad \tilde{R} = -BR^{-1}B^T$$
$$\tilde{A} = A - BR^{-1}S^T C$$

The solution approach used here considers the nonlinearities $\Delta f(x)$ as generating a departure from the linearized H_∞ solution. Therefore, the Lyapunov function $V(x)$ is expressed as

$$V(x) = x^T X x + \Delta V(x) \qquad (9)$$

where X is the solution to Eq. (8). $\Delta V(x)$ will be $\mathcal{O}(x^4)$, and $x^T X x$ will be the Lyapunov function for the linearized solution. Substituting Eq. (9) into the HJI PDE Eq. (4) and using Eqs. (6–8) the following first-order PDE is obtained:

$$0 = 2x^T X \Delta f + \Delta V_x[(\tilde{A} + \tilde{R}X)x + \Delta f] + \tfrac{1}{4}\Delta V_x \tilde{R} \Delta V_x^T \qquad (10)$$

Notice that $\Delta f(x) = 0$ implies that $\Delta V_x = 0$. Stability of the linearized solution implies that all of the eigenvalues of the $\tilde{A} + \tilde{R}X$ are in the left half-plane.

Equation (10) is solved by the method of characteristics.[9] The formulas from Ford[9] apply to a PDE of first order in one dependent variable ($\xi = \Delta V$) and n independent variables ($x_1 \cdots x_n$). Let $p = \Delta V_x^T$, i.e.,

$$p_1 = \frac{\partial \Delta V}{\partial x_1}, \ldots, p_n = \frac{\partial \Delta V}{\partial x_n} \qquad (11)$$

The general first-order PDE has the form

$$\theta(x_1, \ldots, x_n, p_1, \ldots, p_n, \xi) = 0 \qquad (12)$$

The characteristic strips satisfy the following $2n + 1$ differential equations:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \theta_{pi} \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\theta_{x_i} - \theta_\xi p_i \qquad \frac{\mathrm{d}\xi}{\mathrm{d}t} = \theta_{p_1} p_1 + \cdots + \theta_{p_n} p_n \qquad (13)$$

To further simplify notation define $\tilde{F} = \tilde{A} + \tilde{R}X$. Equation (12) becomes

$$\theta(x, p, \xi) = 2x^T X \Delta f + p^T[\tilde{F}x + \Delta f] + p^T(\tilde{R}/4)p = 0 \qquad (14)$$

The characteristic equations for Eq. (14) are (with x replaced by z)

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \left(\frac{\partial \theta(z, p, \xi)}{\partial p}\right)^T = \tilde{F}z + \frac{1}{2}\tilde{R}p + \Delta f(z)$$
$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\left(\frac{\partial \theta(z, p, \xi)}{\partial z}\right)^T - \frac{\partial \theta(z, p, \xi)}{\partial \xi} p$$
$$= -\tilde{F}^T p - 2\Delta f_z^T X z - \Delta f_z^T p - 2X\Delta f(z) \qquad (15)$$
$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = \left[\frac{\partial \theta(z, p, \xi)}{\partial p}\right] p = p^T \tilde{F} z + p^T \Delta f(z) - \frac{1}{2} p^T \tilde{R} p$$

where the independent variable t need not correspond to time. The last scalar equation in Eq. (15) need not be solved unless the term ΔV [Eq. (9)] is needed in computing the Lyapunov function.

Notice that an integral of Eq. (15) is

$$p^T(\tilde{R}/4)p + [\tilde{F}z + \Delta f(z)]^T p + 2z^T X \Delta f(z) = 0 \tag{16}$$

The characteristic equations Eq. (15) can be put into integral form in order to attempt a successive approximation solution procedure. The state and costate equations from Eq. (15) can be written as

$$\begin{bmatrix} \dot{z}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} \tilde{F} & \frac{1}{2}\tilde{R} \\ 0 & -\tilde{F}^T \end{bmatrix} \begin{bmatrix} z(t) \\ p(t) \end{bmatrix} + \begin{bmatrix} \Delta f(z(t)) \\ q(z(t), p(t)) \end{bmatrix} \tag{17}$$

where

$$q(z, p) = -2\Delta f_z^T(z) X z - \Delta f_z^T(z) p - 2X\Delta f(z) \tag{18}$$

and Δf is defined in Eq. (7). The integral of Eq. (17) is

$$\begin{bmatrix} z(t) \\ p(t) \end{bmatrix} = \exp\left\{\begin{bmatrix} \tilde{F} & \frac{1}{2}\tilde{R} \\ 0 & -\tilde{F}^T \end{bmatrix} t\right\} \left\{\begin{bmatrix} z(0) \\ p(0) \end{bmatrix} + \int_0^t \exp\left(-\begin{bmatrix} \tilde{F} & \frac{1}{2}\tilde{R} \\ 0 & -\tilde{F}^T \end{bmatrix}\tau\right) \begin{bmatrix} \Delta f(z) \\ q(z,p) \end{bmatrix} d\tau\right\} \tag{19}$$

Define

$$\Phi(t) = \begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) \\ 0 & \phi_{22}(t) \end{bmatrix} = \exp\left(\begin{bmatrix} \tilde{F} & \frac{1}{2}\tilde{R} \\ 0 & -\tilde{F}^T \end{bmatrix} t\right) \tag{20}$$

Differentiating the ϕ_{ij} yields

$$\dot{\phi}_{11}(t) = \tilde{F}\phi_{11}(t) \qquad \dot{\phi}_{12}(t) = \tilde{F}\phi_{12}(t) + \tfrac{1}{2}\tilde{R}\phi_{22}(t)$$
$$\dot{\phi}_{22}(t) = -\tilde{F}^T\phi_{22}(t) \tag{21}$$

with initial conditions of $\phi_{11}(0) = I$, $\phi_{12}(0) = 0$, $\phi_{22}(0) = I$. Solving for the ϕ_{ij} yields

$$\phi_{11}(t) = e^{\tilde{F}t} \qquad \phi_{22}(t) = e^{-\tilde{F}^T t}$$
$$\phi_{12}(t) = e^{\tilde{F}t}\int_0^t e^{-\tilde{F}\tau}\frac{1}{2}\tilde{R}e^{-\tilde{F}^T\tau}\, d\tau \tag{22}$$

Substituting these into Eq. (19) results in

$$z(t) = \phi_{11}(t)z_0 + \phi_{12}(t)p_0 + \int_0^t [\phi_{11}(t-\tau)\Delta f(\tau) + \phi_{12}(t-\tau)q(\tau)]\, d\tau \tag{23}$$

$$p(t) = \phi_{22}(t)p_0 + \int_0^t \phi_{22}(t-\tau)q(\tau)\, d\tau$$

where the dependency of $q()$ and $\Delta f()$ on z and p has been dropped for notational convenience. Solving for p_0 and substituting this into the $z(t)$ expression yields

$$z(t) = \phi_{11}(t)z_0 + \phi_{12}(t)\phi_{22}^{-1}(t)p(t) + \int_0^t \left[\phi_{11}(t-\tau)\Delta f(\tau) + \left\{\phi_{12}(t-\tau) - \phi_{12}(t)\phi_{22}^{-1}(t)\phi_{22}(t-\tau)\right\}q(\tau)\right] d\tau \tag{24}$$

Define

$$J(t) = \phi_{12}(t)\phi_{22}^{-1}(t) = e^{\tilde{F}t}\int_0^t e^{-\tilde{F}\tau}\frac{1}{2}\tilde{R}e^{-\tilde{F}^T\tau}\, d\tau e^{\tilde{F}^T t}$$
$$= \int_0^t e^{\tilde{F}(t-\tau)}\left(\frac{1}{2}\tilde{R}\right)e^{\tilde{F}^T(t-\tau)}\, d\tau \tag{25}$$

Let $\zeta = t - \tau$. Then,

$$J(t) = \int_0^t e^{\tilde{F}\zeta}\left(\frac{1}{2}\tilde{R}\right)e^{\tilde{F}^T\zeta}\, d\zeta \tag{26}$$

and

$$J = e^{\tilde{F}\zeta}\frac{1}{2}\tilde{R}e^{\tilde{F}^T\zeta}\tilde{F}^{-T}\Big|_0^t - \tilde{F}\int_0^t e^{\tilde{F}\zeta}\frac{1}{2}\tilde{R}e^{\tilde{F}^T\zeta}\, d\zeta\, \tilde{F}^{-T}$$
$$= e^{\tilde{F}t}\tfrac{1}{2}\tilde{R}e^{\tilde{F}^T t}\tilde{F}^{-T} - \tfrac{1}{2}\tilde{R}\tilde{F}^{-T} - \tilde{F}J\tilde{F}^{-T} \tag{27}$$

Note that the integral $J(\infty)$ must exist since $\tilde{F}$ is a stable matrix. Several useful relationships from Eq. (27) are

$$\frac{dJ}{dt} = e^{\tilde{F}t}\left(\frac{1}{2}\tilde{R}\right)e^{\tilde{F}^T t}$$
$$\frac{dJ}{dt} = \tilde{F}J + J\tilde{F}^T + \frac{1}{2}\tilde{R} \qquad J(0) = 0$$
$$\tilde{F}J + J\tilde{F}^T + \tfrac{1}{2}\tilde{R} + e^{\tilde{F}t}\left(-\tfrac{1}{2}\tilde{R}\right)e^{\tilde{F}^T t} = 0 \tag{28}$$
$$\tilde{F}J(\infty) + J(\infty)\tilde{F}^T = -\tfrac{1}{2}\tilde{R}$$

To simplify the integral equation, Eq. (24), define the last term in the integral of Eq. (24) inside the braces as

$$K(t,\tau) = \phi_{12}(t-\tau) - \phi_{12}(t)\phi_{22}^{-1}(\tau) \tag{29}$$

Using Eq. (21), and differentiating, we have

$$\frac{\partial K}{\partial t} = \tilde{F}\phi_{12}(t-\tau) + \frac{1}{2}\tilde{R}\phi_{22}(t-\tau) - \left[\tilde{F}\phi_{12}(t) + \frac{1}{2}\tilde{R}\phi_{22}(t)\right]\phi_{22}^{-1}(\tau)$$
$$= \tilde{F}\left[\phi_{12}(t-\tau) - \phi_{12}(t)\phi_{22}^{-1}(\tau)\right] + \frac{1}{2}\tilde{R}\left[\phi_{22}(t-\tau) - \phi_{22}(t)\phi_{22}^{-1}(\tau)\right]$$
$$= \tilde{F}K(t,\tau) \tag{30}$$

At $t = 0$ we have

$$K(0,\tau) = \phi_{12}(-\tau) - \phi_{12}(0)\phi_{22}^{-1}(\tau) = \phi_{12}(-\tau)$$
$$= e^{-\tilde{F}\tau}\int_0^{-\tau} e^{-\tilde{F}\zeta}\frac{1}{2}\tilde{R}e^{-\tilde{F}^T\zeta}\, d\zeta$$
$$= -e^{-\tilde{F}\tau}\int_0^{\tau} e^{\tilde{F}\mu}\frac{1}{2}\tilde{R}e^{\tilde{F}^T\mu}\, d\mu$$
$$= -e^{-\tilde{F}\tau}J(\tau) \tag{31}$$

Using Eqs. (30) and (31), $K(t,\tau)$ can be expressed as

$$K(t,\tau) = e^{\tilde{F}t}K(0,\tau)$$
$$= -e^{\tilde{F}(t-\tau)}J(\tau)$$
$$= -\phi_{11}(t-\tau)J(\tau) \tag{32}$$

By differentiating Eq. (32), Eq. (30) is obtained. Substituting these expressions into the $z(t)$ equation in Eq. (23) results in

$$z(t) = \phi_{11}(t)z(0) + J(t)p(t) + \int_0^t \phi_{11}(t-\tau)\{\Delta f(\tau) - J(\tau)q(\tau)\}\, d\tau$$
$$= e^{\tilde{F}t}z(0) + J(t)p(t) + e^{\tilde{F}\tau}\int_0^t e^{-\tilde{F}\tau} \times \{\Delta f(\tau) - J(\tau)q(\tau)\}\, d\tau \tag{33}$$

To solve for the costate $p(t)$ in Eq. (23) we have

$$p(t) = e^{-\tilde{F}^T t}p(0) + e^{-\tilde{F}^T t}\int_0^t e^{\tilde{F}^T\tau}q(\tau)\, d\tau \tag{34}$$

Multiplying by $e^{\tilde{F}^T t}$ results in

$$e^{\tilde{F}^T t}p(t) = p(0) + \int_0^t e^{\tilde{F}^T \tau}q(\tau)\,d\tau \quad (35)$$

The boundary condition for $p(t)$ at ∞ is known. Taking the limit as $t \to \infty$ yields

$$0 \cdot (\infty) = p(0) + \int_0^\infty e^{\tilde{F}^T \tau}q(\tau)\,d\tau \quad (36)$$

Solving for $p(0)$ and using the fact that $p(t) \to 0$ as $t \to \infty$ yields

$$p(0) = -\int_0^\infty e^{\tilde{F}^T \tau}q(\tau)\,d\tau \quad (37)$$

Substituting this into Eq. (34) yields the following expression for $p(t)$:

$$\begin{aligned} p(t) &= e^{-\tilde{F}^T t}p(0) + e^{-\tilde{F}^T t}\int_0^t e^{\tilde{F}^T \tau}q(\tau)\,d\tau \\ &= -e^{-\tilde{F}^T t}\left[\int_0^\infty e^{\tilde{F}^T \tau}q(\tau)\,d\tau - \int_0^t e^{\tilde{F}^T \tau}q(\tau)\,d\tau\right] \\ &= -e^{-\tilde{F}^T t}\int_t^\infty e^{\tilde{F}^T \tau}q(\tau)\,d\tau \end{aligned} \quad (38)$$

The resulting integral expressions for the characteristic equations in Eqs. (17) are

$$\begin{aligned} z(t) &= e^{\tilde{F}t}z(0) + J(t)p(t) + e^{\tilde{F}t}\int_0^t e^{-\tilde{F}\tau}\{\Delta f[z(\tau)] \\ &\quad - J(\tau)q[z(\tau), p(\tau)]\}\,d\tau \\ p(t) &= -e^{-\tilde{F}^T t}\int_t^\infty e^{\tilde{F}^T \tau}q[z(\tau), p(\tau)]\,d\tau \end{aligned} \quad (39)$$

Solutions to these equations thus define a curve between $[x(0), p(0)]$ and the equilibrium point (0, 0). It is easy to verify (by differentiating) that these expressions satisfy Eqs. (17). Note that due to the stability of $\tilde{F}$, the integrals in Eq. (39) are defined for only mild restrictions on $\Delta f(x)$.

The successive approximation procedure used to solve the integral expressions in Eq. (39) begins with the solution of the linearized equations,

$$z^{(0)}(t) = e^{\tilde{F}t}z \qquad p^{(0)}(t) = 0 \quad (40)$$

where the superscripts on z and p denote the order of the approximation. These are then substituted into the right-hand side of Eq. (39) to evaluate the next approximation, and so on. After an acceptable degree of approximation has been achieved, the control is generated by evaluating $p(t)$ at $t = 0$. More precisely, the procedure is as follows.

To compute the state feedback control, Eq. (5), $\Delta V_x(x)$ is needed. To obtain $\Delta V_x(x)$ the following three step process is used.

1) Set the initial condition $z(0)$ in Eq. (39) to the current value of x.

2) Generate the desired degree of approximation, starting with Eq. (40).

3) Evaluate $p(0)$ and equate $\Delta V_x(x)$ to $p(0)$.

From Eq. (5) the control is then given by

$$u(x) = [1 \quad 0](-R^{-1})\left\{\tfrac{1}{2}B^T\left(2Xx + \Delta V_x^T\right) + S^T Cx\right\} \quad (41)$$

Local Contraction Mapping

In this section we prove that the successive approximation solution procedure provides a local contraction mapping. This is important in proving the existence and uniqueness of the solution.

Define

$$w(t) = \begin{bmatrix} z(t) \\ p(t) \end{bmatrix}$$

$$q[w(t)] = -2\Delta f_z^T[z(t)]Xz(t) - \Delta f_z^T[z(t)]p(t) - 2X\Delta f[z(t)]$$

$$J(t) = \int_0^t e^{\tilde{F}\tau}\frac{1}{2}\tilde{R}e^{\tilde{F}^T\tau}\,d\tau = e^{\tilde{F}t}\hat{\rho}e^{\tilde{F}^T t} - \hat{\rho}$$

where $\hat{\rho}$ satisfies the Lyapunov equation $\tilde{F}\hat{\rho} + \hat{\rho}\tilde{F}^T = \frac{1}{2}\tilde{R}$. Let

$$s[w(t)] = \Delta f[z(t)] + \hat{\rho}q[w(t)]$$

Note that $q[w]$ and $s[w]$ are second order or higher in w since $\Delta f[w]$ is second order or higher.

The integral equations (39) can be rewritten using the preceding definitions as

$$\begin{aligned} z(t) &= e^{\tilde{F}t}x - J(t)\int_t^\infty e^{\tilde{F}^T(\tau - t)}q[w(\tau)]\,d\tau \\ &\quad + \int_0^t e^{\tilde{F}(t-\tau)}s[w(\tau)]\,d\tau - \int_0^t e^{\tilde{F}t}\hat{\rho}e^{\tilde{F}^T\tau}q[w(\tau)]\,d\tau \\ p(t) &= -\int_t^\infty e^{\tilde{F}^T(\tau - t)}q[w(\tau)]\,d\tau \end{aligned}$$

Let Pw be the successive approximation mapping

$$(Pw)(t) = \begin{bmatrix} e^{\tilde{F}t}x - J(t)\int_t^\infty e^{\tilde{F}^T(\tau-t)}q[w(\tau)]\,d\tau + \int_0^t e^{\tilde{F}(t-\tau)}s[w(\tau)]\,d\tau - \int_0^t e^{\tilde{F}t}\hat{\rho}e^{\tilde{F}^T\tau}q[w(\tau)]\,d\tau \\ -\int_t^\infty e^{\tilde{F}^T(\tau-t)}q[w(\tau)]\,d\tau \end{bmatrix}$$

It is shown that this is a contraction in a sufficiently small neighborhood of the equilibrium $[z = 0, p = 0]$, as follows.

Let $\|\cdot\|_n$ denote the Euclidean norm in R^n. Let $\|\cdot\|_C$ be the norm on $C^{2n}[0, \infty)$, i.e.,

$$\|w(\cdot)\|_C = \sup_{t \in [0,\infty)} \|w(\cdot)\|_{2n}$$

The induced matrix norm of $e^{\tilde{F}t}$ is $\|e^{\tilde{F}t}\|_i = e^{\lambda t}$ where λ is the real part of the rightmost eigenvalue of $\tilde{F}$. Because of the stability of $\tilde{F}$, then $\lambda < 0$. The induced matrix norm of $J(t)$ satisfies

$$\begin{aligned} \|J(t)\|_i &\le \int_0^t \|e^{\tilde{F}\tau}\|_i \left\|\frac{1}{2}\tilde{R}\right\|_i \left\|e^{\tilde{F}^T\tau}\right\|_i d\tau = \int_0^t e^{2\lambda\tau}d\tau \left\|\frac{1}{2}\tilde{R}\right\|_i \\ &= \left(\frac{1 - e^{2\lambda t}}{-\lambda}\right)\left\|\frac{1}{2}\tilde{R}\right\|_i \end{aligned}$$

Now, define

$$B_r = \{y \in R^{2n} : \|y\| \le r\} \qquad w_0(t) = \begin{bmatrix} e^{\tilde{F}t}x \\ 0 \end{bmatrix}$$

$$S_r = \left\{w(\cdot) \in C^{2n}[0, \infty) : \|w(\cdot) - w_0(\cdot)\|_C \le r\right\}$$

Let $w_1(\cdot)$ and $w_2(\cdot)$ be arbitrary elements of S_r; then, $w_1(t)$ and $w_2(t)$ lie in the ball B_r for all $t \in [0, \infty)$,

$$(Pw_1)(t) - (Pw_2)(t)$$

$$= \begin{bmatrix} -J(t)\int_t^\infty e^{\tilde{F}^T(\tau-t)}(q[w_1(\tau)] - q[w_2(\tau)])\,d\tau + \int_0^t e^{\tilde{F}(t-\tau)}(s[w_1(\tau)] - s[w_2(\tau)])\,d\tau - \int_0^t e^{\tilde{F}t}\hat{\rho}e^{\tilde{F}^T\tau}(q[w_1(\tau)] - q[w_2(\tau)])\,d\tau \\ -\int_t^\infty e^{\tilde{F}^T(\tau-t)}(q[w_1(\tau)] - q[w_2(\tau)])\,d\tau \end{bmatrix}$$

Taking the norm on both sides yields

$$\|(Pw_1)(t) - (Pw_2)(t)\|$$

$$\le \|J(t)\|_i \int_t^\infty \left\|e^{\tilde{F}^T(\tau-t)}\right\|_i \kappa_q \|w_1(\tau) - w_2(\tau)\|_{2n}\,d\tau$$

$$+ \int_0^t \left\|e^{\tilde{F}(t-\tau)}\right\|_i \kappa_s \|w_1(\tau) - w_2(\tau)\|_{2n}\,d\tau$$

$$+ \int_0^t \|e^{\tilde{F}t}\|_i \|\hat{\rho}\|_i \left\|e^{\tilde{F}^T\tau}\right\|_i \kappa_q \|w_1(\tau) - w_2(\tau)\|_{2n}\,d\tau$$

$$+ \int_t^\infty \left\|e^{\tilde{F}^T(\tau-t)}\right\|_i \kappa_q \|w_1(\tau) - w_2(\tau)\|_{2n}\,d\tau$$

where κ_s and κ_q are Lipschitz constants in B_r for $s(\cdot)$ and $q(\cdot)$, respectively. These constants are finite due to the second-order nature of $s(\cdot)$ and $q(\cdot)$. Evaluating the matrix norms and using sup norms on $w_1(t) - w_2(t)$ results in

$$\|(Pw_1)(t) - (Pw_2)(t)\| \le \|J(t)\|_i$$

$$\times \left\{\left[\frac{1-e^{2\lambda t}}{-\lambda}\frac{1-e^{\lambda t}}{-\lambda}\left(\frac{1}{2}\|\tilde{R}\|_i + \|\hat{\rho}\|_i\right)\frac{1-e^{\lambda t}}{-\lambda}\right]\kappa_q\right.$$

$$\left. + \frac{1-e^{\lambda t}}{-\lambda}\kappa_s\right\}\|w_1(\cdot) - w_2(\cdot)\|_C$$

$$\le \left[\left(\frac{1}{2\lambda^2}\|\tilde{R}\|_i + \frac{1}{-\lambda}\|\hat{\rho}\|_i\right)\kappa_q + \frac{1}{-\lambda}\kappa_s\right]\|w_1(\cdot) - w_2(\cdot)\|_C$$

Since the expression on the right-hand side does not involve t,

$$\|(Pw_1)(\cdot) - (Pw_2)(\cdot)\|_C \le \rho_r\|w_1(\cdot) - w_2(\cdot)\|_C$$

where

$$\rho_r = \left(\frac{1}{2\lambda^2}\|\tilde{R}\|_i + \frac{1}{-\lambda}\|\hat{\rho}\|_i\right)\kappa_q + \frac{1}{-\lambda}\kappa_s > 0$$

The parameter ρ_r can be made less than one by choosing r small enough because of the second-order nature and Lipschitz continuity of $s(\cdot)$ and $q(\cdot)$. Therefore, P is a contraction mapping on S_r for $r > 0$ sufficiently small.

To show that P maps S_r into itself for r sufficiently small, let $w(\cdot) \in S_r$. Then,

$$(Pw)(t) - w_0(t)$$

$$= \begin{bmatrix} -J(t)\int_t^\infty e^{\tilde{F}^T(\tau-t)}(q[w(\tau)] - q[w_0(\tau)])\,d\tau + \int_0^t e^{\tilde{F}(t-\tau)}(s[w(\tau)] - s[w_0(\tau)])\,d\tau - \int_0^t e^{\tilde{F}t}\hat{\rho}e^{\tilde{F}^T\tau}(q[w(\tau)] - q[w_0(\tau)])\,d\tau \\ -\int_t^\infty e^{\tilde{F}^T(\tau-t)}(q[w(\tau)] - q[w_0(\tau)])\,d\tau \end{bmatrix}$$

$$+ \begin{bmatrix} -J(t)\int_t^\infty e^{\tilde{F}^T(\tau-t)}q[w_0(\tau)]\,d\tau + \int_0^t e^{\tilde{F}(t-\tau)}s[w_0(\tau)]\,d\tau - \int_0^t e^{\tilde{F}t}\hat{\rho}e^{\tilde{F}^T\tau}q[w_0(\tau)]\,d\tau \\ -\int_t^\infty e^{\tilde{F}^T(\tau-t)}q[w_0(\tau)]\,d\tau \end{bmatrix}$$

Taking norms results in

$$\|(Pw)(\cdot) - w_0(\cdot)\|_C \le \rho_r\|w(\cdot) - w_0(\cdot)\|_C + (\|\tilde{R}\|/2\lambda^2)\kappa_q\|x\|_n$$

$$+ (1/-\lambda)\kappa_s\|x\|_n + \|\hat{\rho}\|_i(1/-\lambda)\kappa_q\|x\|_n$$

since

$$\|q[w_0(t)]\|_n \le \|q[e^{\tilde{F}t}x]\|_n \le \kappa_q\|e^{\tilde{F}t}x\|_n \le \kappa_q e^{\lambda t}\|x\|_n \le \kappa_q\|x\|_n$$

and, similarly, for $s[w_0(t)]$. This results in

$$\|(Pw)(\cdot) - w_0(\cdot)\|_C \le \rho_r\|w(\cdot) - w_0(\cdot)\|_C + \left[(\|\tilde{R}\|/2\lambda^2)\kappa_q\right.$$

$$\left. + (1/-\lambda)\kappa_s + \|\hat{\rho}\|_i(1/-\lambda)\kappa_q\right]\|x\|_n$$

where the right-hand side can be made as small as desired by choosing r small.

Missile Nonlinear H_∞ Optimal Control

This section presents the missile dynamics needed for design of the nonlinear H_∞ state feedback control law and outlines the algorithms that are used in the successive approximation solution of the HJI equation. Only a pitch-plane autopilot is considered. The autopilot will command angle of attack by thrust vectoring and deflecting the aerodynamic control surfaces.

The missile's rigid body pitch-plane short period dynamics are described by

$$\dot{\alpha} = (1/V_m)\{\cos(\alpha)(G_z + Z_A + T_z)$$

$$- \sin(\alpha)(G_x + X_A + T_x)\} + q\dot{q} = M_A + M_T \tag{42}$$

where α is the angle of attack, q is the pitch rate, and

$$G_z = g\cos(\theta); \qquad G_x = g\sin(\theta)$$

$$Z_A = Z_\alpha\alpha + Z_\delta\delta_e; \qquad X_A = X_0 + X_\alpha\alpha + X_\delta\delta_e$$

$$M_A = M_0 + M_\alpha\alpha + M_\delta\delta_e + M_q q$$

$$T_z = -(T/m)\sin(\delta_T); \qquad T_x = T/m\cos(\delta_T)$$

$$M_T = -M_{\mathrm{TVC}}\sin(\delta_T)$$

where (G_x, G_z) models gravity, $(X_A Z_A)$ models normalized aerodynamic accelerations (in feet per second squared), M_A models normalized aerodynamic pitching moment (in radians per second squared) $(T_x T_z)$ models thrust forces normalized with respect to the mass (in feet per second squared), and M_T models the pitching moment produced by the thrust vectoring normalized by the pitch inertia (in radians per second squared). The variable δ_e models the pitch fin angle and δ_T models the pitch thrust vector angle.

In addition to these dynamics the integrated aerodynamic/thrust vector control[10] (TVC) system has a common actuator that drives both δ_T and the aerodynamic control surfaces δ_e. These actuator dynamics are modeled using a second-order transfer function.

The autopilot design approach used here is similar to that for linear H_∞ control design.[11] The missile's dynamics are augmented with weighting filter states that will shape the sensitivity, complementary sensitivity, and control activity frequency responses. The performance objectives are to shape the sensitivity $S(s)$ in order to follow AOA commands, to shape the complementary sensitivity $T(s)$ to roll off the plant and to penalize the control activity $C(s)$.

The state vector for this application is given by

$$\boldsymbol{x} = [\alpha, q, \delta, \dot{\delta}, w_S, w_T]^T - \boldsymbol{x}_{\text{trim}}^T$$

where the subscript trim refers to the equilibrium value, and the states w_S and w_T are from first-order weighting filters used in the linear H_∞ design. It is assumed that the aerodynamics are linear. This assumption is used here for convenience but is not necessary for application of the approach. The nonlinearities reside in the dynamics, Eq. (42), and are in the geometric terms in the normal force equation (the $\dot{\alpha}$ equation).

The elements of $\boldsymbol{f}(\boldsymbol{x})$ in Eq. (1) are all linear except the first element, which is

$$f_1 = (1/V_m)\big[-\sin\alpha\big(X_0 + X_\alpha\alpha + X_{\delta_e}\delta_e + (T/m)\big) + \cos\alpha\big(Z_\alpha\alpha + Z_{\delta_e}\delta_e + (T/m)\mu\delta_e\big)\big] + q \tag{43}$$

where $\delta_T = \mu\delta_e$. Let $s\alpha_T$ and $c\alpha_T$ denote the $\sin\alpha_{\text{trim}}$ and $\cos\alpha_{\text{trim}}$, respectively. Expanding f_1 to third-order terms results in

$$\begin{aligned} f_1 &= (1/V_m)[-s\alpha_T X_T + c\alpha_T Z_T] + q_{\text{trim}} \\ &+ (1/V_m)[-s\alpha_T X_\alpha - c\alpha_T X_T + c\alpha_T Z_\alpha - s\alpha_T Z_T]x_1 + x_2 \\ &+ (1/V_m)\big[-s\alpha_T X_{\delta_e} + c\alpha_T\big(Z_{\delta_e} + (T/m)\mu\big)\big]x_3 \\ &+ (1/V_m)\big[-c\alpha_T X_\alpha + \tfrac{1}{2}s\alpha_T X_T - s\alpha_T Z_\alpha - \tfrac{1}{2}c\alpha_T Z_T\big]x_1^2 \\ &+ (1/V_m)\big[-c\alpha_T X_{\delta_e} + s\alpha_T\big(Z_{\delta_e} + (T/m)\mu\big)\big]x_1x_3 \\ &+ (1/V_m)\big[\tfrac{1}{2}s\alpha_T X_\alpha + \tfrac{1}{6}c\alpha_T X_T - \tfrac{1}{2}c\alpha_T Z_\alpha + \tfrac{1}{6}s\alpha_T z_T\big]x_1^3 \\ &+ (1/V_m)\big[\tfrac{1}{2}s\alpha_T X_{\delta_e} - \tfrac{1}{2}c\alpha_T\big(Z_{\delta_e} + (T/m)\mu\big)\big]x_1^2x_3 \end{aligned} \tag{44}$$

where $X_T = X_0 + X_\alpha\alpha_T + X_{\delta_e}\delta_T + (T/m)$ and $Z_T = Z_\alpha\alpha_T + Z_{\delta_e}\delta_T + (T/m)\mu\delta_T$. The first line in this expansion adds to zero because of the definition of trim. The next three lines are linear in the states, and will be denoted as $A_{11}x_1$, $A_{12}x_2$, and $A_{13}x_3$, respectively. The remaining terms represent the higher order nonlinearities $[\mathcal{O}(x^2)]$ and are up to third order. These terms define $\Delta f_1(x)$. Rewriting these terms in a more compact form results in

$$\Delta f_1(x) = c_1x_1^2 + c_2x_1x_3 + c_3x_1^3 + c_4x_1^2x_3 \tag{45}$$

The vector $\boldsymbol{\Delta f}(\boldsymbol{x})$ is given by

$$\boldsymbol{\Delta f}(\boldsymbol{x}) = [\Delta f_1(x), 0, 0, 0, 0, 0]^T \tag{46}$$

and

$$\boldsymbol{\Delta f}_x^T(\boldsymbol{x}) = \begin{bmatrix} 2c_1x_1 + c_2x_3 + 3c_3x_1^2 + 2c_4x_1x_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ c_2x_1 + c_4x_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{47}$$

The next step is to compute the nonlinear part of the control, $\boldsymbol{\Delta V}_x^T(\boldsymbol{x})$. Only the first approximation is computed. Starting with the integral equations, Eq. (39) is

$$\boldsymbol{\Delta V}_x^T(\boldsymbol{x}) = -\int_0^\infty e^{\tilde{F}^T\tau}\big\{-2X\boldsymbol{\Delta f}[z(\tau)] - 2\boldsymbol{\Delta f}_x^T[z(\tau)]Xz(\tau)\big\}\,\mathrm{d}\tau \tag{48}$$

where the stable matrix $\tilde{F}$ and the positive definite symmetric matrix X are provided by an H_∞ design for the linear system, and $z(t)$ is the linear solution of the characteristic equations $z(t) = e^{\tilde{F}t}x$. In this application $\tilde{F}$ has distinct eigenvalues, so that if $\tilde{F}$ is diagonalized as $\tilde{F} = U\Lambda W^T$, then

$$\begin{aligned} e^{\tilde{F}\tau} &= Ue^{\Lambda\tau}W^T = \sum_{i=1}^n \boldsymbol{u}_i\boldsymbol{w}_i^T e^{\lambda_i\tau} \\ e^{\tilde{F}^T\tau} &= \sum_{i=1}^n \boldsymbol{w}_i\boldsymbol{u}_i^T e^{\lambda_i\tau} \end{aligned} \tag{49}$$

where the $\boldsymbol{u}_i$ are right eigenvectors, $\boldsymbol{w}_i$ left eigenvectors, and $\Lambda = \text{diag}[\lambda_i]$ is the diagonal matrix of eigenvalues of $\tilde{F}$. Using Eq. (49), the state x is transformed using $\boldsymbol{y} = W^T\boldsymbol{x}$. This gives

$$z = Ue^{\Lambda\tau}\boldsymbol{y} = \sum_{i=1}^n y_i e^{\lambda_i\tau}\boldsymbol{u}_i \tag{50}$$

and

$$Xz = \sum_{i=1}^n y_i e^{\lambda_i\tau}X\boldsymbol{u}_i \tag{51}$$

The nonlinear expressions for $\Delta f(z)$ and $\Delta f_x^T(z)$ can be written using Eqs. (50) and (51) as the following linear sums of the eigenvalues:

$$\begin{aligned} z_1 &= \sum_{j=1}^n y_j e^{\lambda_j\tau}u_{1j} \\ z_3 &= \sum_{j=1}^n y_j e^{\lambda_j\tau}u_{3j} \\ z_1^2 &= \sum_{j=1}^n\sum_{k=1}^n y_jy_k e^{(\lambda_j+\lambda_k)\tau}u_{1j}u_{1k} \\ z_1z_3 &= \sum_{j=1}^n\sum_{k=1}^n y_jy_k e^{(\lambda_j+\lambda_k)\tau}u_{1j}u_{3k} \\ z_1^3 &= \sum_{j=1}^n\sum_{k=1}^n\sum_{l=1}^n y_jy_ky_l e^{(\lambda_j+\lambda_k+\lambda_l)\tau}u_{1j}u_{1k}u_{1l} \\ z_1^2z_3 &= \sum_{j=1}^n\sum_{k=1}^n\sum_{l=1}^n y_jy_ky_l e^{(\lambda_j+\lambda_k+\lambda_l)\tau}u_{1j}u_{1k}u_{3l} \end{aligned} \tag{52}$$

to obtain

$$\begin{aligned} \boldsymbol{\Delta f} &= \boldsymbol{e}_1\boldsymbol{g}^T z\boldsymbol{e}_1^T z + \boldsymbol{e}_1\boldsymbol{h}^T z\boldsymbol{e}_1^T z\boldsymbol{e}_1^T z \\ \boldsymbol{\Delta f}_z^T &= \big(\boldsymbol{e}_1\boldsymbol{g}^T + \boldsymbol{g}\boldsymbol{e}_1^T\big)z\boldsymbol{e}_1^T + \big(2\boldsymbol{e}_1\boldsymbol{h}^T + \boldsymbol{h}\boldsymbol{e}_1^T\big)z\boldsymbol{e}_1^T z\boldsymbol{e}_1^T \end{aligned} \tag{53}$$

where

$$\begin{aligned} \boldsymbol{e}_1^T &= (1, 0, 0, 0, 0, 0) \\ \boldsymbol{g}^T &= (c_1, 0, c_2, 0, 0, 0) \\ \boldsymbol{h}^T &= (c_3, 0, c_4, 0, 0, 0) \end{aligned} \tag{54}$$

Inserting these and Eq. (49) into Eq. (48) results in

$$\begin{aligned} \boldsymbol{\Delta V}_x^T(\boldsymbol{x}) = W\int_0^\infty &\big[e^{\Lambda\tau}2U\big(\boldsymbol{e}_1\boldsymbol{g}^T + \boldsymbol{g}\boldsymbol{e}_1^T\big)e^{\Lambda\tau}(\boldsymbol{y}\boldsymbol{e}_1XU)e^{\Lambda\tau} \\ &+ e^{\Lambda\tau}\big(2U^TX\boldsymbol{e}_1\boldsymbol{g}^TU\big)e^{\Lambda\tau}\big(\boldsymbol{y}\boldsymbol{e}_1^TU\big)e^{\Lambda\tau} \\ &+ e^{\Lambda\tau}2U^T\big(2\boldsymbol{e}_1\boldsymbol{h}^T + \boldsymbol{h}\boldsymbol{e}_1^T\big)Ue^{\Lambda\tau}\big(\boldsymbol{y}\boldsymbol{e}_1^TXU\big)e^{\Lambda\tau}\big(\boldsymbol{y}\boldsymbol{e}_1^TU\big)e^{\Lambda\tau} \\ &+ e^{\Lambda\tau}\big(2U^TX\boldsymbol{e}_1\boldsymbol{h}^TU\big)e^{\Lambda\tau}\big(\boldsymbol{y}\boldsymbol{e}_1^TU\big)e^{\Lambda\tau}\big(\boldsymbol{y}\boldsymbol{e}_1^TU\big)e^{\Lambda\tau}\big]\,\mathrm{d}\tau\boldsymbol{y} \end{aligned} \tag{55}$$

This can be integrated using the identities

$$\int_a^b [e^{\Lambda\tau}Pe^{\Lambda\tau}Qe^{\Lambda\tau}]_{ij}\,\mathrm{d}\tau = \sum_k \frac{P_{ik}Q_{kj}}{\lambda_i+\lambda_j+\lambda_k} \times \{\exp[(\lambda_i+\lambda_j+\lambda_k)b] - \exp[(\lambda_i+\lambda_j+\lambda_k)a]\} \tag{56}$$

$$\int_a^b [e^{\Lambda\tau}Pe^{\Lambda\tau}Qe^{\Lambda\tau}R]_{ij}\,\mathrm{d}\tau = \sum_k \sum_l \frac{P_{ik}Q_{kl}R_{lj}}{\lambda_i+\lambda_j+\lambda_k+\lambda_l} \times \{\exp[(\lambda_i+\lambda_j+\lambda_k+\lambda_l)b] - \exp[(\lambda_i+\lambda_j+\lambda_k+\lambda_l)a]\}$$

to obtain

$$\Delta V_x^T(x) = WMy \tag{57}$$

where

$$M_{ij} = -\sum_k S^{(3)}_{ijk} y_k - \sum_k \sum_l S^{(4)}_{ijkl} y_k y_l \tag{58}$$

with

$$S^{(3)}_{ijk} = \frac{\left[2\left(U^Te_1\right)_i(U^Tg)_k + 2(U^Tg)_i\left(U^Te_1\right)_k\right]\left(U^TXe_1\right)_j + 2\left(U^TXe_1\right)_i(U^Tg)_k\left(U^Te_1\right)_j}{\lambda_i+\lambda_j+\lambda_k}$$

$$S^{(4)}_{ijkl} = \frac{[\eta]\left(U^TXe_1\right)_l\left(U^Te_1\right)_j + 2\left(U^TXe_1\right)_i(U^Th)_k\left(U^Te_1\right)_l\left(U^Te_1\right)_j}{\lambda_i+\lambda_j+\lambda_k+\lambda_l} \tag{59}$$

where

$$\eta = 4\left(U^Te_1\right)_i(U^Th)_k + 2(U^Th)_i\left(U^Te_1\right)_k \tag{60}$$

Defining $T^{(3)}$ and $T^{(4)}$ arrays by

$$T^{(3)}_{jk} = \sum_i w_i S^{(3)}_{ijk} \qquad T^{(4)}_{jkl} = \sum_i w_i S^{(4)}_{ijkl} \tag{61}$$

results in a polynomial expression in the transformed states y. The gradient of the Lyapunov function $\Delta V_x^T(x)$ becomes

$$\Delta V_x^T(x) = \sum_j \sum_k T^{(3)}_{jk} y_j y_k + \sum_j \sum_k \sum_l T^{(4)}_{jkl} y_j y_k y_l \tag{62}$$

The coefficients $T^{(3)}_{jk}$ form a $6 \times 6 \times 6$ array and $T^{(4)}_{jkl}$ a $6 \times 6 \times 6 \times 6$ array, and can be calculated off line.

Simulation Results

This section presents the simulation results of the nonlinear H_∞ control law. In forming the nonlinear H_∞ control law, several assumptions were introduced in modeling the nonlinearities. It is important to present these modeling assumptions so that it is clear what nonlinearities are present in this problem.

It is assumed that the aerodynamics are linear and are modeled at a fixed AOA. This assumption is used here for convenience but is not necessary for application of the approach. The nonlinearities reside in the dynamics, Eq. (42), and are in the geometric terms in the normal force equation (the $\dot{\alpha}$ equation).

The TVC deflection $\sin(\delta_{\mathrm{TVC}})$ in the $\dot{\alpha}$ equation has been replaced by $\mu\delta_e$. This linearity assumption for the actuator deflection is appropriate here because the TVC actuator is mechanically limited to 10-deg deflection. The actual TVC flow angle does not deflect the same amount as the nozzle angle due to losses in the nozzle. The losses further limit the actual vector angle, reducing it to approximately 8.5 deg (in this application). Over this limited range the $\sin(\delta_{\mathrm{TVC}})$ is linear.

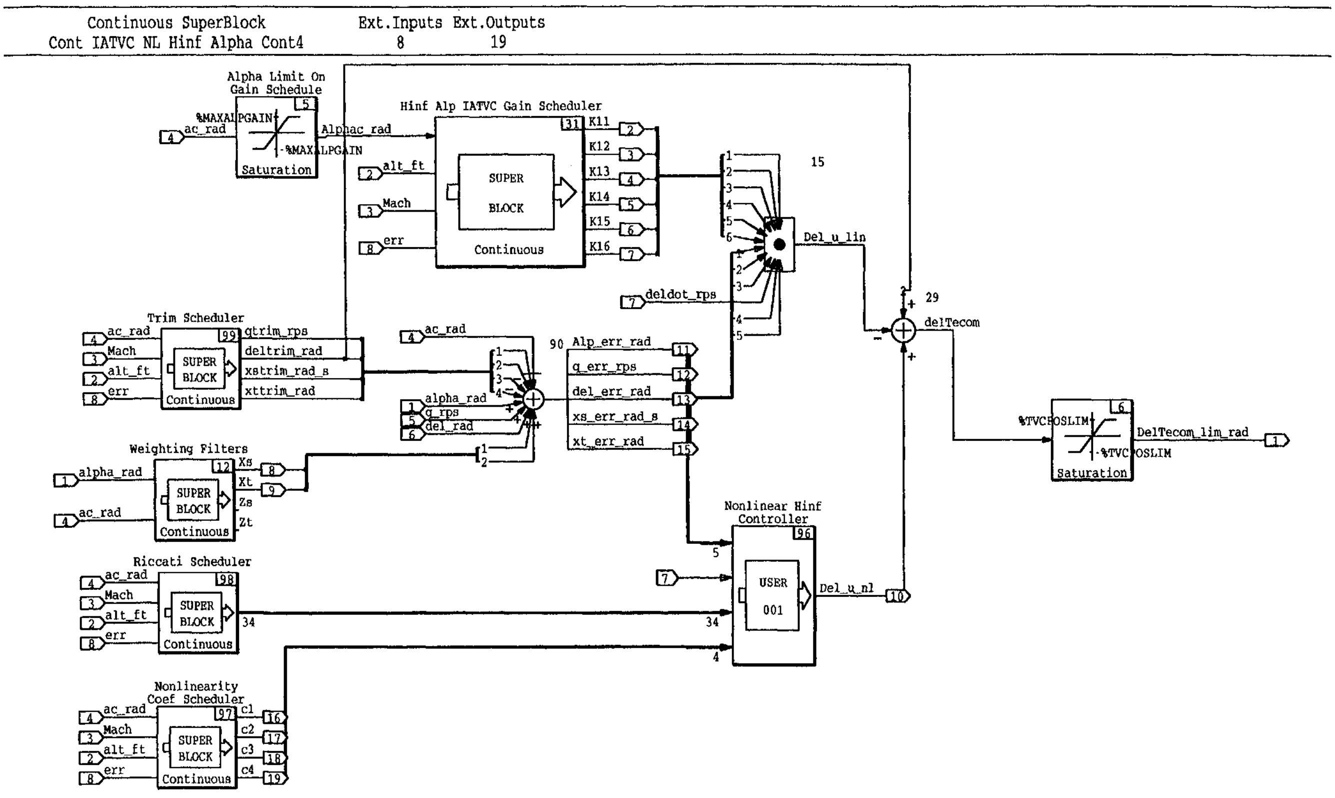

Fig. 2 MATRIXx implementation of the autopilot.

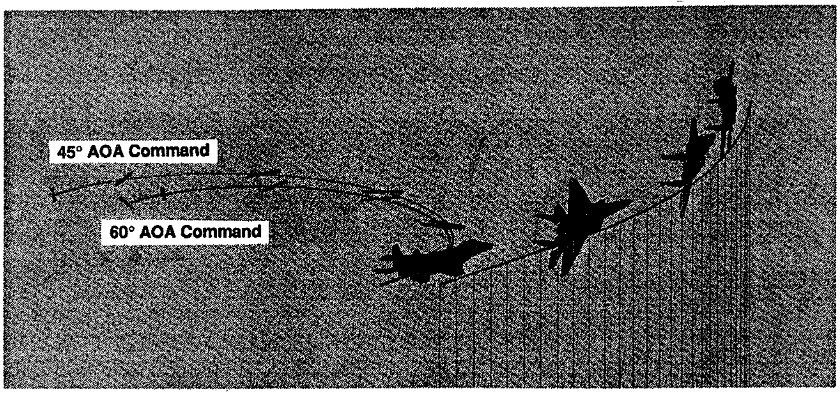

Fig. 3 Animation of agile turn to the rear hemisphere.

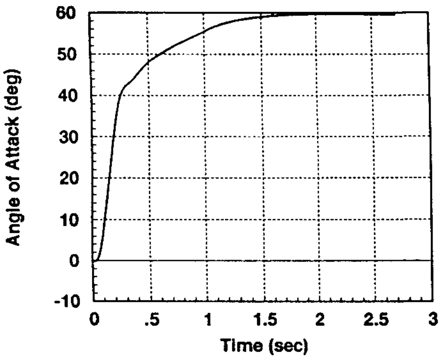

Fig. 4 Angle-of-attack time history.

The nonlinearity f_1 described in Eq. (43) is modeled using a third-order polynomial. This additional modeling assumption adequately captures the nonlinearities introduced by the $\sin(\alpha)$ and $\cos(\alpha)$. Since the aerodynamic parameters are held fixed at a constant AOA, the $\sin(\alpha)$ and $\cos(\alpha)$ are the dominant nonlinearities in this problem. During the agile missile turn, the AOA ranges from 0 to 60 deg. This range significantly exercises the $\sin(\alpha)$ and $\cos(\alpha)$. Modeling the nonlinearities using polynomials is an important aspect of our solution approach. The polynomial models make it significantly easier to solve the integral expressions for the characteristic equations.

The next step is to compute the nonlinear part of the control, $\Delta V_x^T(x)$. Only the first approximation is computed in these results. Computing the first approximation requires solving the integral in Eq. (48), where the stable matrix $\tilde{F}$ and the positive definite symmetric matrix X are provided by an H_∞ design for the linear system, and $z(t)$ is the linear solution of the characteristic equations $z(t) = e^{\tilde{F}t}x$.

In this application $\tilde{F}$ has distinct eigenvalues, so that if $\tilde{F}$ is diagonalized as $\tilde{F} = U\Lambda W^T$, then the integral expression in Eq. (4.3) of Ref. 7 can be solved by representing the matrix exponential with its modal expansion. This makes this integral expression a combination of polynomials weighted by exponentials, which are easy to integrate.

The gradient of the Lyapunov function $\Delta V_x^T(x)$ is solved for using Eq. (62). Figure 1 illustrates the calculations used in computing $\Delta V_x^T(x)$. Fortran subroutines were developed to implement these algorithms. Note that a linear gain schedule is calculated off line and stored for use in computing the linear part of the control law. The linear gain schedule was computed at the design points shown in Table 1.

Figure 2 shows our MATRIXx implementation of the nonlinear H_∞ control law. The super block has eight inputs [α, altitude, Mach number, α command, pitch rate, TVC nozzle deflection, TVC nozzle rate, and fuel ratio (err: 1 = full, 0 = empty)]. The fuel ratio is used to schedule the gains with changes in the center of gravity. The control is the TVC nozzle deflection command computed by the autopilot. This command has three terms: the trim deflection, the linear H_∞ contribution, and the nonlinear H_∞ contribution. The trim deflection is calculated in the trim scheduler superblock. The linear H_∞ contribution is calculated by looking up the linear gains in the Hinf Alp IATVC gain scheduler superblock, shown in the figure. The nonlinear H_∞ software described in Fig. 1 is implemented in the user code block User 001.

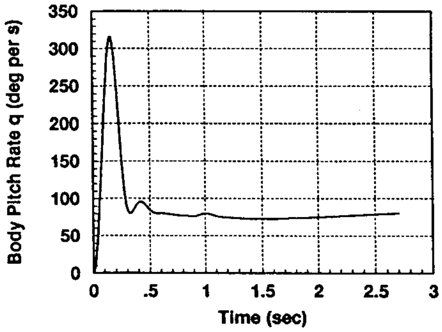

Fig. 5 Pitch rate time history.

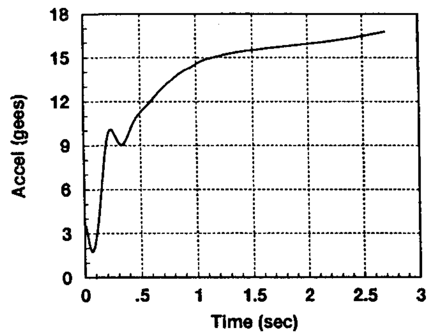

Fig. 6 Normal acceleration time history.

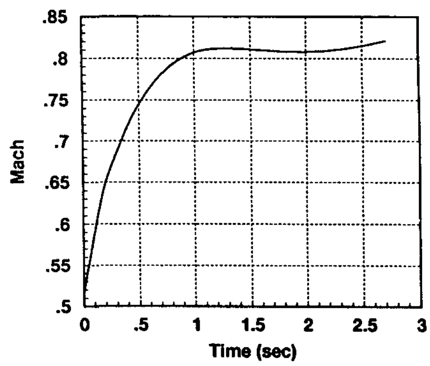

Fig. 7 Mach number time history.

Figure 3 shows an animation of the simulation results. The figure shows an F-15 launch of the agile missile with missile trajectories for a 45- and 60-deg α command. The simulation stops when the missile's heading has changed by 180 deg. The goal is to perform this heading change maneuver as quickly as possible.

Figures 4–10 show time histories of important simulation variables. The α response follows the command as desired. The actuator

Table 1 Linear autopilot design points

AOA, deg	Mach	Alt, kft	C.G.
−100	0.1	0	0
−90	0.6	10	1
−80	0.8	35	
−70	1.0	—	—
−60	1.15	—	—
−50	1.5	—	—
−40	2.0	—	—
−30	3.0	—	—
−20	5.0	—	—
−10	—	—	—
−5	—	—	—
0	—	—	—
5	—	—	—
10	—	—	—
20	—	—	—
30	—	—	—
40	—	—	—
50	—	—	—
60	—	—	—
70	—	—	—
80	—	—	—
90	—	—	—
100	—	—	—

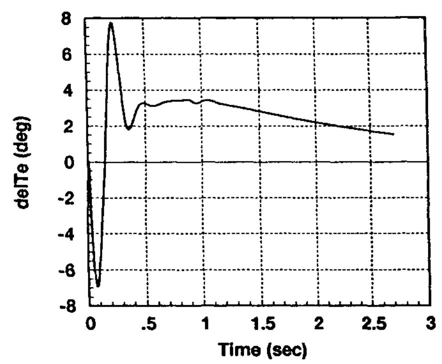

Fig. 8 TVC nozzle deflection time history.

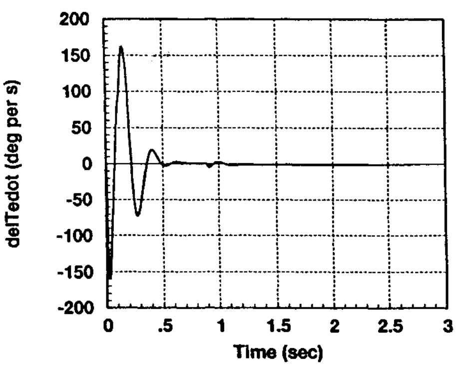

Fig. 9 TVC nozzle rate time history.

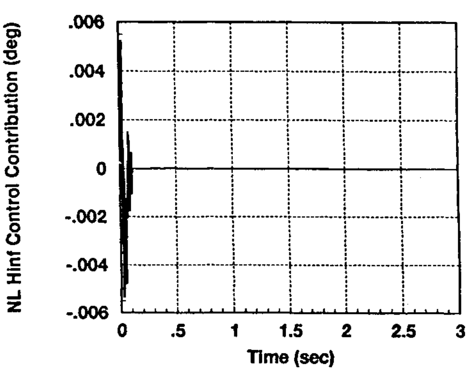

Fig. 10 Nonlinear contribution to the control.

time histories are well within the deflection and rate limits for the actuator.

Figure 10 shows the contribution to the control made by solving for the nonlinear part $\Delta V_x^T(x)$. This contribution is small when compared to the linear control. Since the linear gains are scheduled every 10-deg AOA, the nonlinear aspect of the $\sin(\alpha)$ and $\cos(\alpha)$ are captured in the linear gain Table 1. This leads to a possible conclusion that if the linear design adequately covers the nonlinearities, then the contribution to the control by $\Delta V_x^T(x)$ will be small. This is the case here. The linear aerodynamic model used significantly reduced the nonlinear nature of this flight control problem.

Conclusions and Future Research

A solution approach was presented for solving the HJI partial differential equation that arises in nonlinear H_∞ optimal control problems. The solution approach using successive approximation was applied to a missile flight control problem with six state variables. Although somewhat complicated, the state feedback control algorithms are implementable using standard software techniques. This solution approach has produced reasonable algorithms for solving the nonlinear partial differential equations that arise in the study of nonlinear H_∞ optimal control problems.

Nonlinear simulation (based on a linear aerodynamic model of the missile) was used to test the algorithms and resulted in good performance. TVC nozzle deflections and rates commanded by the autopilot are well within limits. The angle-of-attack time history tracked the command very well.

Future research in this area will focus on developing new software for implementing the successive approximation solution procedure and replacing the linear aerodynamic model with nonlinear aerodynamics. The new software will be used to calculate any number of successive approximations (only the first was used in this study). When this software is complete the linear assumption used in modeling the missile's aerodynamics will be removed, and this agile turn to the rear hemisphere study using nonlinear aerodynamics will be repeated to better test the nonlinear control. It is expected that when the linear aerodynamic model is replaced by the nonlinear aerodynamic model the nonlinear contribution to the control will be more significant.

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